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Non-commutative superstring worldsheet

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Abstract. In this paper we consider the worldsheet of the superstring as a non-commutative space. Some additional terms can be added to the superstring action, such that for an ordinary worldsheet they are zero. The expansion of this extended action up to the first order of the non-commutativity parameter leads to the new supersymmetric action for the string. For the closed superstring, we obtain the boundary state that describes a brane. From the open string point of view, the new boundary conditions on the worldsheet bosons generalize the non-commutativity of spacetime. Finally, we suggest some definitions for the non-commutativity parameter of the superstring worldsheet.

1 Introduction

Non-commutative geometry plays a fascinating role in string theory. There has been a great deal of interest recently in non-commutative theories, stimulated by their connection with string theory and M-theory; for a review and comprehensive list of references see [1] and [2–8]. The idea that the coordinates of spacetime do not commute at sufficiently small distance scales is related to the non-perturbative backgrounds of string and M-theories [2–5, 8]. Non-commutativity on D-branes in the presence of a constant background B-field was the original motivation [2–4]. The worldvolume of a D-brane with a constant background B-field is a simple and concrete example of a non-commutative spacetime, in which gauge and matter fields live [2, 6].

Non-commutative field theories have rich structures. The embedding of these theories into string theory [3], suggests that these structures may be directly relevant to reconsidering the familiar notions of the superstrings and the low energy limits of them. In other words, any change in the string theory affects the whole non-commutativity. Now we introduce some of these changes.

We consider the worldsheet of the superstring as a two dimensional non-commutative space. Therefore we can introduce some additional terms to the superstring action so that for the ordinary worldsheet they are zero. For the small non-commutativity parameter of the string worldsheet we develop the worldsheet supersymmetry for this action. The boundary conditions of an open string with non-commutative worldsheet lead to the generalized non-commutativity parameter of spacetime. In this case the non-commutativity of spacetime is a consequence of the B-field and the non-commutativity of the string worldsheet. The closed string emitted from a brane with background field has a boundary state that is generalized by the non-commutativity of its worldsheet.

Our motivation for studying non-commutativity of the string worldsheet is the following. If the worldsheet lives in a non-commutative spacetime, it is natural to expect it to inherit the non-commutativity from the spacetime. This can be seen from the fact that the pull-back of the spacetime non-commutativity parameter on the string worldsheet is not zero.

It is worth emphasizing that such theory is inherently non-conformal. The parameter of non-commutativity introduces a length scale in the worldsheet which breaks the scale invariance and subsequently the conformal invariance of the theory. Despite lack of conformal invariance, for the following reasons we shall investigate the model.

From the renormalization group and flows it is shown that a large (small) distance of the spacetime corresponds to a small (large) distance of the worldsheet. In other words we have the relation $L^2 = \ln(\Lambda/\mu)$, where Λ^{-1} is a characteristic two dimensional distance that is very much shorter than the two dimensional distance μ^{-1} that for the worldsheet is seen and L is a characteristic spacetime distance [9]. In fact Λ is a two dimensional UV cut-off. Now consider a finite UV cut-off. This will certainly break the scale invariance of the worldsheet theory. If we allow the scale invariance of the worldsheet to be broken at very short distances on the worldsheet, we can interpret the worldsheet non-commutativity parameter as the UV cut-off for the worldsheet.

This paper is organized as follows. In Sect. 2, we briefly review the superstring with an ordinary worldsheet. In Sect. 3, we present a new action and the corresponding supersymmetry for the superstring with a non-commutative worldsheet. In Sect. 4, we study the closed string and its boundary state, which describes a brane. In Sect. 5, we obtain the boundary conditions of the open string, in the presence of a brane. In Sect. 6, some definitions for the

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non-commutativity parameter of the worldsheet of the superstring is suggested.

2 Superstring with ordinary worldsheet

A superstring in the presence of a brane with background fields is described by the action [10]

$$S = -\frac{1}{4\pi\alpha'}\int_{\Sigma} \mathrm{d}^2\xi(\sqrt{-h}h^{ab}g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + \epsilon^{ab}B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu - \mathrm{i}\sqrt{-h}g_{\mu\nu}\bar{\psi}^\mu \rho^a \partial_a \psi^\nu) + \frac{1}{4\pi\alpha'}\int_{\partial\Sigma} \mathrm{d}\zeta F_{\alpha\beta}\left(X^\alpha \partial_\zeta X^\beta + \frac{\mathrm{i}}{2}\theta^\alpha\theta^\beta\right), \quad (1)$$

where Σ is the worldsheet of the string, and $\partial\Sigma$ is its boundary. The indices $\alpha, \beta, \gamma, \cdots$, show the brane directions. The coordinate ζ is tangent to the boundary of the string worldsheet. The field $B_{\mu\nu}$ is the NS⊗NS massless field, and $F_{\alpha\beta}$ is the constant field strength of a $U(1)$ gauge field A_α. The field θ^μ is the following combination of the components ψ_1^μ and ψ_2^μ of the worldsheet fermion ψ^μ,

$$\theta^\mu = \psi_1^\mu + \mathrm{i}\psi_2^\mu. \quad (2)$$

Let $F_{\alpha\beta} = 0$, $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, \cdots, 1)$ and $B_{\mu\nu}$ be the constant background field. Also consider $h_{ab} = \eta_{ab} = \mathrm{diag}(-1, 1)$. Then the equations of motion are

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0,$$
$$\partial_+\psi_1^\mu = 0,$$
$$\partial_-\psi_2^\mu = 0, \quad (3)$$

where $\partial_\pm = (1/2)(\partial_\tau \pm \partial_\sigma)$. The invariance of the action under the worldsheet supersymmetry transformations

$$\delta X^\mu = \bar{\epsilon}\psi^\mu,$$
$$\delta\psi^\mu = -\mathrm{i}\rho^a\partial_a X^\mu \epsilon \quad (4)$$

leads to the following boundary state equations for the closed superstring:

$$(\partial_\tau X^\alpha - B^\alpha{}_\beta \partial_\sigma X^\beta)_{\tau_0} \mid B\rangle = 0, \quad (5)$$

$$(\partial_\sigma X^i)_{\tau_0} \mid B\rangle = 0 \quad (6)$$

for the bosonic part, and

$$(\psi_1^\alpha - \mathrm{i}\psi_2^\alpha + B^\alpha{}_\beta(\psi_1^\beta + \mathrm{i}\psi_2^\beta))_{\tau_0} \mid B\rangle = 0, \quad (7)$$

$$(\psi_1^i + \mathrm{i}\psi_2^i)_{\tau_0} \mid B\rangle = 0 \quad (8)$$

for the fermionic part. The indices $i, j, \cdots$, show the transverse directions of the brane. Since the presence of the brane breaks half of the supersymmetry, for deriving (5)–(8) we used the relation $\epsilon_2 = \mathrm{i}\epsilon_1$.

The boundary conditions of the open superstring are

$$(\partial_\sigma X^\alpha - B^\alpha{}_\beta \partial_\tau X^\beta)_{\sigma_0} = 0,$$
$$(\partial_\tau X^i)_{\sigma_0} = 0,$$
$$(\psi_1^\alpha + \mathrm{i}\psi_2^\alpha + B^\alpha{}_\beta(\psi_1^\beta - \mathrm{i}\psi_2^\beta))_{\sigma_0} = 0,$$
$$(\psi_1^i - \mathrm{i}\psi_2^i)_{\sigma_0} = 0, \quad (9)$$

where the boundaries are at $\sigma_0 = 0, \pi$.

3 Non-commutative worldsheet

Let ξ^0 and ξ^1 be the coordinates of the worldsheet of the superstring. The "star product" between two arbitrary functions $f(\xi^0, \xi^1)$ and $g(\xi^0, \xi^1)$ is

$$f(\xi^0, \xi^1) * g(\xi^0, \xi^1) = \exp\left(\frac{\mathrm{i}}{2}\theta^{ab}\frac{\partial}{\partial\zeta^a}\frac{\partial}{\partial\eta^b}\right) f(\zeta^0, \zeta^1)g(\eta^0, \eta^1) \mid_{\zeta=\eta=\xi} . \quad (10)$$

Therefore, there is non-commutativity between ξ^0 and ξ^1, i.e.

$$\xi^a * \xi^b - \xi^b * \xi^a = \mathrm{i}\theta^{ab}. \quad (11)$$

Later we shall discuss the antisymmetric tensor θ^{ab}.

For the coordinates (σ, τ) let η_{ab} be the metric of the string worldsheet, then the superstring action under the star product becomes

$$S^* = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma(g_{\mu\nu}\partial_a X^\mu * \partial^a X^\nu + \epsilon^{ab}B_{\mu\nu}\partial_a X^\mu * \partial_b X^\nu - \mathrm{i}g_{\mu\nu}\bar{\psi}^\mu * \rho^a\partial_a\psi^\nu) + \bar{S}^*, \quad (12)$$

where $\epsilon^{01} = -\epsilon^{10} = 1$. The action $\bar{S}^*$ contains the bosonic and the fermionic fields of the worldsheet, and when the star product changes to the usual product, i.e. for $\theta^{ab} = 0$, it vanishes. We consider $\bar{S}^*$ as in the following:

$$\bar{S}^* = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\left(C_{\mu\nu}X^\mu * X^\nu + k^{ab}A_{\mu\nu}\partial_a X^\mu * \partial_b X^\nu + \frac{\mathrm{i}}{2}\epsilon^{ab}S_{\mu\nu}\bar{\psi}^\mu * \rho_a\rho_b\psi^\nu\right), \quad (13)$$

where $C_{\mu\nu}$ and $A_{\mu\nu}$ are arbitrary antisymmetric tensors and $S_{\mu\nu}$ and k^{ab} are arbitrary symmetric tensors. Many other terms can be considered such that their usual product vanish. For example the terms

$$C_{\mu\nu}\partial_{a_1}\cdots\partial_{a_m}X^\mu * \partial^{a_1}\cdots\partial^{a_m}X^\nu,$$
$$k^{ab}A_{\mu\nu}\partial_{a_1}\cdots\partial_{a_l}\partial_a X^\mu * \partial^{a_1}\cdots\partial^{a_l}\partial_b X^\nu,$$
$$\epsilon^{ab}S_{\mu\nu}\partial_{a_1}\cdots\partial_{a_n}\bar{\psi}^\mu * \rho_a\rho_b\partial^{a_1}\cdots\partial^{a_n}\psi^\nu \quad (14)$$

are zero for the usual product. The arbitrary numbers m, l and n are positive integers. Because of the derivatives, we do not introduce these terms to the action (13). After expanding in terms of the powers of θ^{ab}, the first non-zero term of the second term of the action (13) contains derivatives of order four; for simplification this term is neglected too. Also there are other terms such as $S^{(1)}_{\mu\nu}\psi_1^\mu * \psi_1^\nu$ and $S^{(2)}_{\mu\nu}\psi_2^\mu * \psi_2^\nu$ and their derivatives like (14), that for symmetric matrices $S^{(1)}_{\mu\nu}$ and $S^{(2)}_{\mu\nu}$ vanish under the usual product. These terms do not have worldsheet covariant forms, therefore we also put them away.

Now we consider the expansion of the action (12) up to the first order of θ^{ab} and study closed and open superstrings of it:

$$S^* = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\left(g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu + \epsilon^{ab}B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right.$$

$$
\begin{aligned}
&- \mathrm{i}g_{\mu\nu}\bar{\psi}^{\mu}\rho^{a}\partial_{a}\psi^{\nu} + \frac{1}{2}\theta^{ab}g_{\mu\nu}\partial_{a}\bar{\psi}^{\mu}\rho^{a'}\partial_{a'}\partial_{b}\psi^{\nu} \\
&- \frac{1}{4}\epsilon^{a'b'}\theta^{ab}S_{\mu\nu}\partial_{a}\bar{\psi}^{\mu}\rho_{a'}\rho_{b'}\partial_{b}\psi^{\nu} \\
&+ \left.\frac{\mathrm{i}}{2}\theta^{ab}C_{\mu\nu}\partial_{a}X^{\mu}\partial_{b}X^{\nu}\right) + \mathcal{O}(\theta^{2}). \qquad (15)
\end{aligned}
$$

The second term and the last three terms are total derivatives. Note that θ^{ab} has only one independent component; therefore it can be written as

$$\theta^{ab} = \theta\epsilon^{ab}. \qquad (16)$$

In the coordinate system (σ, τ) we choose θ as a constant parameter.

Let us define $B'_{\mu\nu}$ as follows:

$$B'_{\mu\nu} = B_{\mu\nu} + \frac{i}{2}\theta C_{\mu\nu}. \qquad (17)$$

Therefore the B-term and C-term of the action (15) can be combined to a B'-term. If we assume $C_{\mu\nu}$ to be a linear combination of $B_{\mu\nu}$ and $F_{\mu\nu}$ (field strength of a $U(1)$ gauge field)

$$C_{\mu\nu} = aF_{\mu\nu} + bB_{\mu\nu}, \qquad (18)$$

gauge invariance of $B'_{\mu\nu}$ under the gauge transformations

$$
\begin{aligned}
&A_{\mu} \to A_{\mu} + \Lambda_{\mu}, \\
&B_{\mu\nu} \to B_{\mu\nu} + \partial_{\mu}\Lambda_{\nu} - \partial_{\nu}\Lambda_{\mu}, \qquad (19)
\end{aligned}
$$

requires the following relation between coefficients "a" and "b" and the parameter θ:

$$2 + \mathrm{i}\theta(a + b) = 0. \qquad (20)$$

This equation implies $a \neq -b$, which means if $B'_{\mu\nu}$ is a gauge invariant field, that $C_{\mu\nu}$ in the form of combination (18) is not gauge invariant.

From now on we neglect $\mathcal{O}(\theta^2)$ in the action (15). Let $S_{\mu\nu}$ and $C_{\mu\nu}$ be constant, i.e. independent of the spacetime coordinates. We introduce the new supersymmetry transformations,

$$
\begin{aligned}
&\delta X^{\mu} = \bar{\epsilon}\psi^{\mu} - \mathrm{i}\theta S^{\mu}{}_{\nu}\partial_{\tau}(\bar{\epsilon}\psi^{\nu}), \\
&\delta\psi^{\mu} = -\mathrm{i}\rho^{a}\partial_{a}X^{\mu}\epsilon. \qquad (21)
\end{aligned}
$$

These transformations form a closed algebra. To see this, consider two successive transformations with supersymmetry parameters ϵ and ϵ'; then

$$
\begin{aligned}
[\delta_{\epsilon}, \delta_{\epsilon'}]X^{\mu} &= \delta_{\epsilon}(\delta_{\epsilon'}X^{\mu}) - (\epsilon \leftrightarrow \epsilon') \\
&= 2\mathrm{i}\bar{\epsilon}\rho^{a}\epsilon'(\partial_{a}X^{\mu} - \mathrm{i}\theta S^{\mu}{}_{\nu}\partial_{\tau}\partial_{a}X^{\nu}) \qquad (22)
\end{aligned}
$$

for the worldsheet bosons, and

$$[\delta_{\epsilon}, \delta_{\epsilon'}]\psi^{\mu} = 2\mathrm{i}\bar{\epsilon}\rho^{a}\epsilon'(\partial_{a}\psi^{\mu} - \mathrm{i}\theta S^{\mu}{}_{\nu}\partial_{\tau}\partial_{a}\psi^{\nu}) \qquad (23)$$

for the worldsheet fermions. To obtain the last equation, one should use the equation of motion of ψ^{μ}, which is $\rho^{a}\partial_{a}\psi^{\mu} = 0$.

As has been mentioned, the presence of a brane breaks half of the supersymmetry. For $\epsilon_2 = \mathrm{i}\epsilon_1 \equiv \mathrm{i}\varepsilon$, the above transformations become

$$
\begin{aligned}
&\delta X^{\mu} = -\varepsilon(\theta^{\mu} - \mathrm{i}\theta S^{\mu}{}_{\nu}\partial_{\tau}\theta^{\nu}), \\
&\delta\psi_{1}^{\mu} = -2\mathrm{i}\varepsilon\partial_{-}X^{\mu}, \\
&\delta\psi_{2}^{\mu} = 2\varepsilon\partial_{+}X^{\mu}. \qquad (24)
\end{aligned}
$$

We shall use these transformations to obtain the boundary conditions of superstrings.

4 Closed superstring

For the closed superstring let the metric $g_{\mu\nu}$ be $\eta_{\mu\nu}$. Now we concentrate to the R⊗R and the NS⊗NS sectors of the type II superstring. These sectors imply that the surface terms of the variation of the action (15) vanish. This variation gives the boundary state equations for the closed superstring, emitted from the brane, as

$$(\partial_{\tau}X^{\alpha} - B'^{\alpha}{}_{\beta}\partial_{\sigma}X^{\beta} - B'^{\alpha}{}_{i}\partial_{\sigma}X^{i})_{\tau_0} \mid B\rangle = 0, \quad (25)$$

$$(\delta X^{i})_{\tau_0} \mid B\rangle = 0 \quad (26)$$

for the bosonic part. Equation (26) implies that $\partial_{\sigma}X^{i}$ vanishes on the boundary, and will be dropped from (25). From now on we assume that the mixed components of $S_{\mu\nu}$ are zero, i.e.

$$S_{i\alpha} = 0. \qquad (27)$$

According to the supersymmetry transformations (24) and the bosonic part of the boundary state equations, i.e. (25) and (26), there are the following boundary state equations for the worldsheet fermions:

$$(\psi_{1}^{i} + \mathrm{i}\psi_{2}^{i} - \mathrm{i}\theta S^{i}{}_{j}\partial_{\tau}(\psi_{1}^{j} + \mathrm{i}\psi_{2}^{j}))_{\tau_0} \mid B\rangle = 0, \qquad (28)$$

$$
\begin{aligned}
&(\psi_{1}^{\alpha} - \mathrm{i}\psi_{2}^{\alpha} + B'^{\alpha}{}_{\beta}(\psi_{1}^{\beta} + \mathrm{i}\psi_{2}^{\beta}) + \mathrm{i}\theta S^{\alpha}{}_{\beta}\partial_{\tau}(\psi_{1}^{\beta} - \mathrm{i}\psi_{2}^{\beta}) \\
&\quad -\mathrm{i}\theta B'^{\alpha}{}_{\beta}S^{\beta}{}_{\gamma}\partial_{\tau}(\psi_{1}^{\gamma} + \mathrm{i}\psi_{2}^{\gamma}))_{\tau_0} \mid B\rangle = 0. \qquad (29)
\end{aligned}
$$

As expected, these equations respect the supersymmetry transformations. We explicitly show this. That is, from the fermionic boundary conditions (28) and (29) and the supersymmetry transformations, we obtain the bosonic boundary conditions (25) and (26). Equation (28) and the first transformation of (24) give the transverse bosonic boundary condition (26).

To see the consistence of (25) and (29), let us write the supersymmetry transformations of the left and the right moving parts of X^{μ}

$$
\begin{aligned}
&\delta X_{\mathrm{L}}^{\mu} = -\mathrm{i}\varepsilon(\psi_{2}^{\mu} - \theta S^{\mu}{}_{\nu}\partial_{\tau}\psi_{1}^{\nu}), \\
&\delta X_{\mathrm{R}}^{\mu} = -\varepsilon(\psi_{1}^{\mu} + \theta S^{\mu}{}_{\nu}\partial_{\tau}\psi_{2}^{\nu}). \qquad (30)
\end{aligned}
$$

The sum of these transformations gives δX^{μ} of (24). The difference of these gives

$$
\begin{aligned}
\delta X'^{\mu} &= \delta X_{\mathrm{L}}^{\mu} - \delta X_{\mathrm{R}}^{\mu} \\
&= \varepsilon(\lambda^{\mu} + \mathrm{i}\theta S^{\mu}{}_{\nu}\partial_{\tau}\lambda^{\nu}), \qquad (31)
\end{aligned}
$$

where λ^μ is

$$\lambda^\mu \equiv \psi_1^\mu - \mathrm{i}\psi_2^\mu. \tag{32}$$

From (29) we have

$$(\varepsilon(\lambda^\alpha + \mathrm{i}\theta S^\alpha{}_\beta \partial_\tau \lambda^\beta) - B'^\alpha{}_\beta[-\varepsilon(\theta^\beta - \mathrm{i}\theta S^\beta{}_\gamma \partial_\tau \theta^\gamma)])_{\tau_0} \mid B\rangle = 0. \tag{33}$$

According to the transformations (24) and (31) this is

$$(\delta X'^\alpha - B'^\alpha{}_\beta \delta X^\beta)_{\tau_0} \mid B\rangle = 0, \tag{34}$$

which is equivalent to the equation

$$(\partial_\sigma X'^\alpha - B'^\alpha{}_\beta \partial_\sigma X^\beta)_{\tau_0} \mid B\rangle = 0. \tag{35}$$

For the coordinate X'^μ we have the relation

$$\partial_\sigma X'^\mu = \partial_\tau X^\mu, \tag{36}$$

that can be seen from the solution of the equation of motion,

$$X_\mathrm{L}^\mu = x_\mathrm{L}^\mu + 2\alpha' p_\mathrm{L}^\mu(\tau+\sigma) + \frac{\mathrm{i}}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu \mathrm{e}^{-2\mathrm{i}n(\tau+\sigma)},$$
$$X_\mathrm{R}^\mu = x_\mathrm{R}^\mu + 2\alpha' p_\mathrm{R}^\mu(\tau-\sigma) + \frac{\mathrm{i}}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu \mathrm{e}^{-2\mathrm{i}n(\tau-\sigma)}, \tag{37}$$

where $X^\mu = X_\mathrm{L}^\mu + X_\mathrm{R}^\mu$ and $X'^\mu = X_\mathrm{L}^\mu - X_\mathrm{R}^\mu$. Therefore (35) and (36) give the boundary state equation (25). Now we obtain the boundary state $\mid B\rangle$.

4.1 Boundary state

Combining the solutions of the equations of motion and boundary state equations, we obtain these equations in terms of modes. Consider some of the brane directions and some of the transverse directions of the brane to be compact on tori.

The boundary state of the bosonic part is

$$\mid B_\mathrm{bos}, \tau_0\rangle = \sum_{\{p^\alpha\}} \mid B_\mathrm{bos}, \tau_0, p^\alpha\rangle, \tag{38}$$

where

$$\mid B_\mathrm{bos}, \tau_0, p^\alpha\rangle = \frac{T_p}{2}\sqrt{\det(1+B')}\exp\left(\mathrm{i}\alpha'\tau_0\sum_i (p_{op}^i)^2\right) \times \delta^{(9-p)}(x^i - y^i)\exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\mathrm{e}^{4\mathrm{i}m\tau_0}\alpha_{-m}^\mu \Phi_{\mu\nu}\tilde{\alpha}_{-m}^\nu\right) \times \mid 0\rangle \prod_i \mid p_\mathrm{L}^i = p_\mathrm{R}^i = 0\rangle \prod_\alpha \mid p^\alpha\rangle. \tag{39}$$

The set $\{y^i\}$ shows the position of the brane. The orthogonal matrix $\Phi^\mu{}_\nu$ is

$$\Phi^\mu{}_\nu = (Q^\alpha{}_\beta, -\delta^i{}_j),$$
$$Q^\alpha{}_\beta = [(1+B')^{-1}(1-B')]^\alpha{}_\beta. \tag{40}$$

The state (39) is the general form of the state of [11]. The momentum of the closed string along the compact directions of the brane, i.e. $\{X^{\alpha_c}\}$, is

$$p^{\alpha_c} = \frac{1}{\alpha'}B'^{\alpha_c}{}_{\beta_c}L^{\beta_c},$$
$$L^{\beta_c} = N^{\beta_c}R^{\beta_c}, \tag{41}$$

where R^{β_c} is the radius of compactification of the X^{β_c}-direction and N^{β_c} is the winding number of the closed string around the X^{β_c}-direction. For the interpretation of (41) see [11].

For the NS⊗NS sector, we have the following fermionic boundary state equations:

$$\left((1-2r\theta S)^i{}_j b_r^j \mathrm{e}^{-2\mathrm{i}r\tau_0} + \mathrm{i}(1+2r\theta S)^i{}_j \tilde{b}_{-r}^j \mathrm{e}^{2\mathrm{i}r\tau_0}\right) \mid B_f, \tau_0\rangle_\mathrm{NS} = 0, \tag{42}$$

for the transverse directions of the brane. For the directions along the brane we have

$$\left([1+B'+2r\theta(1-B')S]^\alpha{}_\beta\, b_r^\beta \mathrm{e}^{-2\mathrm{i}r\tau_0} - \mathrm{i}[1-B'-2r\theta(1+B')S]^\alpha{}_\beta\, \tilde{b}_{-r}^\beta \mathrm{e}^{2\mathrm{i}r\tau_0}\right) \mid B_f, \tau_0\rangle_\mathrm{NS} = 0. \tag{43}$$

In both of these equations, "r" is a negative or positive half-integer number.

Equations (42) and (43) have the following solution:

$$|B_f, \eta, \tau_0\rangle_\mathrm{NS} = K_\mathrm{NS}\exp\left[\mathrm{i}\eta\sum_{r=1/2}^{\infty}(\mathrm{e}^{4\mathrm{i}r\tau_0} b_{-r}^\mu \Phi_{\mu\nu}^{(r)}\tilde{b}_{-r}^\nu)\right] \mid 0\rangle, \tag{44}$$

where $\eta = \pm 1$ is introduced to make the GSO projection easily. The matrix $\Phi_{\mu\nu}^{(r)}$ has the definition

$$\Phi^\mu_{(r)\ \nu} = (\Lambda^\alpha_{(r)\ \beta}, -H^i_{(r)\ j}), \quad r \geq \frac{1}{2}, \tag{45}$$
$$H^i_{(r)\ j} = [(1-2r\theta S)^{-1}(1+2r\theta S)]^i{}_j, \tag{46}$$
$$\Lambda^\alpha_{(r)\ \beta} = \Big([1+B'+2r\theta(1-B')S]^{-1} \times [1-B'-2r\theta(1+B')S]\Big)^\alpha{}_\beta. \tag{47}$$

The consistence of the solutions of (43) for positive and negative "r" requires the following relation between $B'^\alpha{}_\beta$ and $S^\alpha{}_\beta$,

$$B'^\alpha{}_\beta(S^2)^\beta{}_\gamma = (S^2)^\alpha{}_\beta B'^\beta{}_\gamma. \tag{48}$$

That is, B' and S^2 should commute. This is a restriction that naturally arises on C and S.

The factor K_NS is expected by the path integral with boundary action

$$K_\mathrm{NS} = \prod_{r=1/2}^{\infty}(\det[1+B'+2r\theta(1-B')S]^\alpha{}_\beta). \tag{49}$$

This is the general form of the result of [12]. For the ordinary worldsheet i.e. $\theta = 0$, this reduces to "1", as expected (note that $\sum_{r=1/2}^{\infty} 1 \leftrightarrow \lim_{t\to 0}(2^t - 1)\zeta(t) = 0$). The assumption of the smallness of θ gives

$$K_{\rm NS} = 1 + \frac{\theta}{24}{\rm Tr}[(Q_0 S)^{\alpha}{}_{\beta}] + \mathcal{O}(\theta^2), \tag{50}$$

where Q_0 is given by (40) for $\theta = 0$. Note that we made use of $\sum_{r=1/2}^{\infty} r \leftrightarrow -(1/2)\zeta(-1) = 1/24$, and

$$\det(1 + \theta M) = 1 + \theta {\rm Tr}(M) + \mathcal{O}(\theta^2) \tag{51}$$

for the matrix M to obtain (50). Up to the first order of θ, C does not appear in $K_{\rm NS}$.

For the R⊗R sector, the boundary state equations of the worldsheet fermions in terms of the modes are

$$(d_0^i + {\rm i}\tilde{d}_0^i) \mid B_f, \tau_0\rangle_{\rm R} = 0, \tag{52}$$

$$((1 - 2n\theta S)^i{}_j\, d_n^j {\rm e}^{-2{\rm i}n\tau_0} + {\rm i}(1 + 2n\theta S)^i{}_j\, \tilde{d}_{-n}^j {\rm e}^{2{\rm i}n\tau_0}) \mid B_f, \tau_0\rangle_{\rm R} = 0 \tag{53}$$

for the transverse directions of the brane, and

$$(d_0^{\alpha} - {\rm i}Q^{\alpha}{}_{\beta}\tilde{d}_0^{\beta}) \mid B_f, \tau_0\rangle_{\rm R} = 0, \tag{54}$$

$$\begin{aligned}&([1 + B' + 2n\theta(1 - B')S]^{\alpha}{}_{\beta}\, d_n^{\beta}{\rm e}^{-2{\rm i}n\tau_0}\\ &- {\rm i}[1 - B' - 2n\theta(1 + B')S]^{\alpha}{}_{\beta}\, \tilde{d}_{-n}^{\beta}{\rm e}^{2{\rm i}n\tau_0}) \mid B_f, \tau_0\rangle_{\rm R}\\ &= 0\end{aligned} \tag{55}$$

for the brane directions. In (53) and (55) the number "n" is a non-zero integer.

The solution of (52)–(55) is

$$\begin{aligned}&\mid B_f, \eta, \tau_0\rangle_{\rm R}\\ &= K_{\rm R} \exp\left[{\rm i}\eta \sum_{n=1}^{\infty}({\rm e}^{4{\rm i}n\tau_0} d_{-n}^{\mu}\Phi_{\mu\nu}^{(n)}\tilde{d}_{-n}^{\nu})\right] \mid B_f, \eta\rangle_{\rm R}^{(0)},\end{aligned} \tag{56}$$

where $\mid B_f, \eta\rangle_{\rm R}^{(0)}$ is the solution of (52) and (54) [13,14]. We have

$$\mid B_f, \eta\rangle_{\rm R}^{(0)} = \mathcal{M}_{AB}^{(\eta)} \mid A\rangle \mid \tilde{B}\rangle, \tag{57}$$

where $\mid A\rangle$ and $\mid \tilde{B}\rangle$ describe the vacuum of the fermionic zero modes d_0^{μ} and $\tilde{d}_0^{\mu}$. The matrix $\mathcal{M}^{(\eta)}$ is [13,15]

$$\mathcal{M}^{(\eta)} = \bar{C}\Gamma^0\Gamma^{\bar{\alpha}_1}\cdots\Gamma^{\bar{\alpha}_p}\left(\frac{1 + {\rm i}\eta\Gamma_{11}}{1 + {\rm i}\eta}\right) \times \exp\left(-\frac{1}{2}B'_{\alpha\beta}\Gamma^{\alpha}\Gamma^{\beta}\right), \tag{58}$$

where "$\bar{C}$" is the charge conjugation matrix. Also, the brane is along the directions $\{X^{\bar{\alpha}_1}, \cdots, X^{\bar{\alpha}_p}\}$. Note that for the exponential in (58) there is a convention: the exponential must be expanded with the convention that all gamma matrices anticommute; therefore, there are a finite number of terms.

Again consistence of the solutions of (55), for positive and negative "n" leads to the condition (48).

For the R⊗R sector of the superstring the matrices $\Phi_{(n)}$, $H_{(n)}$ and $\Lambda_{(n)}$ are

$$\Phi^{\mu}_{(n)\ \nu} = (\Lambda^{\alpha}_{(n)\ \beta},\ -H^i_{(n)\ j}), \quad n \geq 1, \tag{59}$$

$$H^i_{(n)\ j} = [(1 - 2n\theta S)^{-1}(1 + 2n\theta S)]^i{}_j, \tag{60}$$

$$\Lambda^{\alpha}_{(n)\ \beta} = \Big([1 + B' + 2n\theta(1 - B')S]^{-1} \times [1 - B' - 2n\theta(1 + B')S]\Big)^{\alpha}{}_{\beta}. \tag{61}$$

The factor $K_{\rm R}$ is

$$K_{\rm R} = \prod_{n=1}^{\infty}(\det[1 + B' + 2n\theta(1 - B')S]^{\alpha}{}_{\beta}). \tag{62}$$

For the ordinary worldsheet, this factor reduces to the expected result $(\det[(1 + B)^{\alpha}{}_{\beta}])^{-1/2}$ of [12] (note that $\sum_{n=1}^{\infty} 1 \leftrightarrow \zeta(0) = -1/2$). The parameter θ is small, therefore

$$\begin{aligned}K_{\rm R} &= \frac{1}{\sqrt{\det[(1 + B)^{\alpha}{}_{\beta}]}}\\ &\times \left[1 - \frac{\theta}{2}\left(\frac{\rm i}{2}{\rm Tr}[(1 + B)^{-1}C]^{\alpha}{}_{\beta} + \frac{1}{3}{\rm Tr}[(Q_0 S)^{\alpha}{}_{\beta}]\right)\right]\\ &+ \mathcal{O}(\theta^2),\end{aligned} \tag{63}$$

where we have used $\sum_{n=1}^{\infty} n \leftrightarrow \zeta(-1) = -1/12$.

5 Open superstring

Now we obtain the boundary conditions of the open superstring. From now on consider the metric of the spacetime to be constant, $g_{\mu\nu}$. Also let the mixed components of the metric be zero, i.e. $g_{\alpha j} = 0$. Furthermore assume that the field B' has non-zero components only along the brane, i.e. the components B'_{ij} and $B'_{\alpha j}$ are zero. The variation of the action (15) gives the boundary conditions

$$(\delta X^i)_{\sigma_0} = 0, \tag{64}$$

$$(g_{\alpha\beta}\partial_{\sigma}X^{\beta} - B'_{\alpha\beta}\partial_{\tau}X^{\beta})_{\sigma_0} = 0 \tag{65}$$

for the bosonic part, where $\sigma_0 = 0, \pi$ show the boundaries. The worldsheet fermions obey the following boundary conditions:

$$(g_{ij}(\psi_1^j - {\rm i}\psi_2^j) - {\rm i}\theta S_{ij}\partial_{\tau}(\psi_1^j - {\rm i}\psi_2^j))_{\sigma_0} = 0, \tag{66}$$

$$\Big(g_{\alpha\beta}(\psi_1^{\beta} + {\rm i}\psi_2^{\beta}) + B'_{\alpha\beta}(\psi_1^{\beta} - {\rm i}\psi_2^{\beta}) + {\rm i}\theta S_{\alpha\beta}\partial_{\tau}(\psi_1^{\beta} + {\rm i}\psi_2^{\beta}) - {\rm i}\theta B'_{\alpha\beta}S^{\beta}{}_{\gamma}\partial_{\tau}(\psi_1^{\gamma} - {\rm i}\psi_2^{\gamma})\Big)_{\sigma_0} = 0. \tag{67}$$

Similar to the closed superstring, one can show that these boundary conditions respect the worldsheet supersymmetry. The open string boundary conditions (64)–(67) can be obtained from the closed one, with the exchanges $\partial_\tau X^\mu \leftrightarrow \partial_\sigma X^\mu$ and $\psi_1^\mu \to -\psi_1^\mu$. This is equivalent to the change $\epsilon_2 \to -\epsilon_2$ in the supersymmetry transformations (21).

According to the boundary condition (64), the transverse directions of the brane remain ordinary. The boundary condition (65) says that the worldvolume of the brane is a non-commutative space. The parameter of the spacetime non-commutativity is [2]

$$\Theta^{\mu\nu} = -2\pi\alpha' \left(\frac{1}{g+B'} B' \frac{1}{g-B'}\right)^{\mu\nu}. \tag{68}$$

The appearance of B' instead of B in this quantity shows the effects of the non-commutativity of the worldsheet on the spacetime non-commutativity. Thus for non-zero "θ" and "C", the brane directions are non-commutative, even if the B-field vanishes.

If we apply the assumption of the smallness of θ in (68), we obtain

$$\Theta^{\mu\nu} = \Theta_0^{\mu\nu} + \frac{\mathrm{i}}{2}\theta\Omega^{\mu\nu} + \mathcal{O}(\theta^2), \tag{69}$$

where the matrix Ω is

$$\begin{aligned}\Omega = &\Theta_0 C(g-B)^{-1} - (g+B)^{-1}C\Theta_0 \\ &- 2\pi\alpha'(g+B)^{-1}C(g-B)^{-1};\end{aligned} \tag{70}$$

as expected, Ω is an antisymmetric matrix. The parameters $\Theta_0^{\mu\nu}$ show the spacetime non-commutativity for the ordinary string worldsheet.

The effective metric of the open string is [2]

$$\begin{aligned}G_{\mu\nu} &= g_{\mu\nu} - (B'g^{-1}B')_{\mu\nu} = G_{\mu\nu}^{(0)} \\ &\quad - \frac{\mathrm{i}}{2}\theta(Bg^{-1}C + Cg^{-1}B)_{\mu\nu} + \frac{1}{4}\theta^2(Cg^{-1}C)_{\mu\nu}, \\ G^{\mu\nu} &= \left(\frac{1}{g+B'} g \frac{1}{g-B'}\right)^{\mu\nu}.\end{aligned} \tag{71}$$

Up to the order θ, $G^{\mu\nu}$ is

$$\begin{aligned}G^{\mu\nu} &= G_0^{\mu\nu} + \frac{\mathrm{i}}{2}\theta(G_0C(g-B)^{-1} - (g+B)^{-1}CG_0)^{\mu\nu} \\ &\quad + \mathcal{O}(\theta^2),\end{aligned} \tag{72}$$

where $G_0^{\mu\nu}$ and $G_{\mu\nu}^{(0)}$ refer to the metric that is seen by the open string with ordinary worldsheet.

Now we use the metric (71) to calculate the first correction of Yang–Mills and open string couplings

$$\frac{1}{g_{\mathrm{YM}}^2} = \frac{(\alpha')^{(3-p)/2}}{(2\pi)^{p-2}g_s}\left(\frac{\det(g+B')}{\det G}\right)^{1/2}, \tag{73}$$

$$G_s = \frac{(\alpha')^{(3-p)/2}}{(2\pi)^{p-2}} g_{\mathrm{YM}}^2. \tag{74}$$

These give

$$g_{\mathrm{YM}} = g_{\mathrm{YM}}^{(0)}\left(1 + \frac{\mathrm{i}}{8}\theta \mathrm{Tr}[(g+B)^{-1}C]\right) + \mathcal{O}(\theta^2), \tag{75}$$

$$G_s = G_s^{(0)}\left(1 + \frac{\mathrm{i}}{4}\theta \mathrm{Tr}[(g+B)^{-1}C]\right) + \mathcal{O}(\theta^2), \tag{76}$$

where $g_{\mathrm{YM}}^{(0)}$ and $G_s^{(0)}$ are the Yang–Mills and open string couplings for the ordinary worldsheet, in non-commutative spacetime.

6 The parameter of the non-commutativity

Now we suggest some definitions for the non-commutativity parameter of the string worldsheet. These definitions are independent of the assumption of the smallness of θ^{ab}, that we used in previous sections. If we change the worldsheet coordinates ξ^0 and ξ^1 to $\xi'^0 = \xi'^0(\xi^0, \xi^1)$ and $\xi'^1 = \xi'^1(\xi^0, \xi^1)$, the tensor θ^{ab} changes to $\theta'^{a'b'}$,

$$\theta'^{a'b'} = \frac{\partial \xi'^{a'}}{\partial \xi^a}\frac{\partial \xi'^{b'}}{\partial \xi^b}\theta^{ab}. \tag{77}$$

As expected, this implies that non-commutativity depends on the coordinate system of the string worldsheet. Note that according to the relation (77) we can choose a coordinate system on the string worldsheet with constant non-commutativity parameter, i.e. independent of the worldsheet coordinates.

6.1 The first definition

Since the quantity $\theta_{ab}\theta^{ab}$ does not change from one coordinate system of the worldsheet to another one, we give the first definition of θ^{ab} as

$$\theta_{ab}\theta^{ab} = \Theta_{\mu\nu}\Theta^{\mu\nu}. \tag{78}$$

In the coordinate system (σ, τ), the left hand side is $-2\theta^2$. Raising the indices of $\Theta_{\mu\nu}$ leads to

$$\theta^2 = \frac{1}{2}\Theta^{\mu\nu}G_{\nu\nu'}\Theta^{\nu'\mu'}G_{\mu'\mu} \equiv \frac{1}{2}\mathrm{Tr}(\Theta G \Theta G). \tag{79}$$

Again for $C_{\mu\nu} \neq 0$, the right hand side also contains θ. Therefore (79) is an equation for θ.

6.2 The second definition

Consider the non-commutative Yang–Mills theory and the background dependence of it [2]. For the background B, the non-commutativity of spacetime is described by Θ_0, and for B', it is described by Θ. It has been discussed in [2] that the background independence of non-commutative Yang–Mills at fixed "$g_{\mu\nu}$" leads to this fact: that the quantity $g_{\mathrm{YM}}^2(\det\Theta)^{1/2}$ must be invariant under the changes of the background field. Therefore

$$g_{\mathrm{YM}}^2\sqrt{\det\Theta} = (g_{\mathrm{YM}}^{(0)})^2\sqrt{\det\Theta_0}. \tag{80}$$

We suggest this equation as a second definition for the parameter θ. According to (68) and (73), the left hand side is a function of θ; therefore from this equation θ is calculated. Note that for the zero slope limit [2], the above equation reduces to an identity, i.e. the left hand side will be independent of θ.

7 Conclusions and remarks

The non-commutative worldsheet of the superstring affects many things. The additional terms to the non-commutative action of the superstring generalize the supersymmetry transformations, the boundary state of the closed superstring, the boundary conditions of the open superstring, Yang–Mills and open string couplings, and many other things. The new closed string boundary state describes a brane that is more general than the mixed branes [11,15]. The non-commutativity of the string worldsheet also changes the spacetime non-commutativity. Therefore even if the background B-field vanishes, spacetime remains non-commutative.

We suggested some definitions for the non-commutativity parameter of the string worldsheet that relate this parameter to the corresponding one of spacetime.

As it has been discussed in [9], renormalization exhibits the large distance spacetime physics to be encoded in the short distance structure of the worldsheet. In other words, the renormalization is justified by the divergence of $L^2 = \ln(\Lambda/\mu)$.

According to the renormalization group, the short two dimensional UV cut-off distance Λ^{-1} slides to shorter and shorter distances Λ_0^{-1}. In other words, there is an effective worldsheet at the two dimensional distance Λ^{-1}. The effective worldsheet can be used to calculate effective string effects at spacetime distance L. Since the non-commutativity parameter of the worldsheet breaks the scale invariance of the worldsheet theory, it can be interpreted as the UV cut-off. Therefore the cut-off distances Λ_0^{-1} and Λ^{-1} correspond to two non-commutativity parameters θ_0 and θ respectively. This implies that for the non-commutativity parameter of the worldsheet, there are some bounds.

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References

1. M.R. Douglas, N.A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2002) hep-th/0106048
2. N. Seiberg, E. Witten, JHEP **9908**, 032 (1999), hep-th/9908142
3. A. Connes, M.R. Douglas, A. Schwarz, JHEP **9802**, 003 (1998) hep-th/9711162
4. Y.E. Cheung, M. Krogh, Nucl. Phys. B **528**, 185 (1998), hep-th/9803031; A. Schwarz, Nucl. Phys. B **534**, 720 (1998), hep-th/9805034; C. Hofman, E. Verlinde, JHEP **9812**, 010 (1998), hep-th/9810116
5. M.R. Douglas, C. Hull, JHEP **9802**, 008 (1998), hep-th/9711165
6. E. Fradkin, A. Tseytlin, Phys. Lett. B **158**, 316 (1985); F. Ardalan, H. Arfaei, M.M. Sheikh-Jabbari, JHEP **9902**, 016 (1999) hep-th/9810072; C.S. Chu, P.M. Ho, Nucl. Phys. B **550**, 151 (1999) hep-th/9812219; Nucl. Phys. B **568**, 447 (2000), hep-th/9906192; V. Schomerus, JHEP **9906**, 030 (1999), hep-th/9903205
7. A. Fayyazuddin, M. Zabzine, Phys. Rev. D **62**, 046004 (2000) hep-th/9911018; P.M. Ho, Y.T. Yeh, Phys. Rev. Lett. **85**, 5523 (2000) hep-th/0005159; D. Bigatti, L. Susskind, Phys. Rev. D **62**, 066004 (2000) hep-th/9908056; F. Ardalan, H. Arfaei, M.M. Sheikh-Jabbari, Nucl. Phys. B **576**, 578 (2000), hep-th/9906161; N. Ishibashi, hep-th/9909176; D. Kamani, Mod. Phys. Lett. A **17**, 237 (2002) hep-th/0107184; Europhys. Lett. **57**, 672 (2002), hep-th/0112153
8. E. Witten, Nucl. Phys. B **460**, 335 (1996)
9. D. Friedan, A tentative theory of large distance physics, hep-th/0204131
10. R.G. Leigh, Mod. Phys. Lett. A 4 **28**, 2767 (1989)
11. H. Arfaei, D. Kamani, Phys. Lett. B **452**, 54 (1999), hep-th/9909167
12. C.G. Callan, C. Lovelace, C.R. Nappi, S.A. Yost, Nucl. Phys. B **308**, 221 (1988); Nucl. Phys. B **288**, 525 (1987)
13. M. Billo, P. Di Vecchia, M. Frau, A. Lerda, I. Pesando, R. Russo, S. Sciuto, Nucl. Phys. B **526**, 199 (1998), hep-th/9802088
14. P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda, R. Russo, Nucl. Phys. B **507**, 259 (1997), hep-th/9707068
15. H. Arfaei, D. Kamani, Nucl. Phys. B **561**, 57 (1999), hep-th/9911146; Phys. Lett. B **475**, 39 (2000), hep-th/9909079; D. Kamani, Phys. Lett. B **487**, 187 (2000), hep-th/0010019